Stochastic Modeling: Exercise Class 3.

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Exercise 1. Assume one wants to simulate the uniform distribution over the unit ball B_1 in $\mathbb{R}^d$.

- Show that d random variables $U_1, \dots, U_d \sim \mathcal{U}(0,1)$ iid follows the uniform distribution over $[0,1]^d$.
- Deduce a uniform random vector over the hyper-rectangle $[-1,1]^d$ from $U_1, \dots, U_d$.
- Show that the rejection step in order to simulate the uniform distribution over B_1 has efficiency $M = 2^d/|B_1|$.
- Describe the behavior of M with respect to the dimension d.

Exercise 2. Give the constants a, b_- and b_+ for the ratio algorithm for the following densities h (up to constants)

- Cauchy $h(x) = \frac{1}{1+x^2}$,
- Exponential $h(x) = e^{-x}$, $x > 0$,
- Gaussian $h(x) = e^{-x^2/2}$,

and for each case compute the efficiency of the associated rejection step.

Exercise 3. Consider the integral $I = \int_\Delta h(\mathbf{x})d\mathbf{x}$ where $\Delta \subset \mathbb{R}^d$ and $|\Delta| < \infty$.

- Show that from d random variables $U_1, \dots, U_d$ one can simulate a uniform distribution $\mathbf{U}$ over a hyper-rectangle R so that $\Delta \subseteq R$.
- Show that $I_n = \frac{1}{n}\sum_{i=1}^n g(\mathbf{U}_i)$ where $g(\mathbf{x}) = |R|h(\mathbf{x})\mathbb{1}_{\{\mathbf{x}\in\Delta\}}$ is an unbiased approximation of I.
- Decompose the variance $\mathrm{Var}\,(g(\mathbf{U}))$ into two terms, one increasing with the efficiency M of the rejection step and another one independent of $M = |R|/|\Delta|$.
- We assume that h is positive and bounded over Δ. Consider the set

$$\mathcal{D} = \Big\{(\mathbf{x}, y) \in \mathbb{R}^{d+1} : \mathbf{x} \in \Delta, 0 \le y \le h(\mathbf{x})\Big\}$$

  and the random variable $Y = \mathbb{1}_{\mathbf{V}\in\mathcal{D}}$ where $\mathbf{V} \sim \mathcal{U}(R \times [0, \sup_\Delta h])$. Determine the distribution of Y.
- Show that $I'_n = \frac{|R|\sup_\Delta h}{n}\sum_{i=1}^n Y_i$ is an alternative unbiased approximation of I, $Y_1, \dots Y_n$ iid versions of Y.
- Compute the variance of Y.
- Compare the variances associated to I_n and I'_n. Which method has the best accuracy?